JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 2, March–April 1995

Reduced-Order Nonlinear Analysis of Aircraft Dynamics

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Results from nonlinear dynamical systems theory are used in understanding flight dynamics at large angles of attack and sideslip. For flight in this regime traditional methods of stability analysis based on linear systems theory are of limited use. This is especially so in the identification of postcritical behavior. Instabilities encountered at large angles of attack and sideslip such as pitch-up, nose-slice, and wing-rock are readily examined within the context of bifurcation theory. Examples of nonlinear analysis using reduced-order models and a combined symbolic/numerical computational approach are discussed and the robustness of the bifurcations to unmodeled dynamics is examined.

Nomenclature

b, c	= wing span and chord length
g	= acceleration due to gravity
h	= altitude
I_x, I_y, I_z, I_{xz}	= mass moments of inertia
L, D, Y	= lift, drag, and side force
l, m, n	= roll, pitch, and yaw moments
M	= Mach number
P, Q, R	= normalized rotational rates
p, q, r	= roll, pitch, and yaw velocities
q_T	= dynamic pressure
S	= reference wing area
V_T	= true air speed
W	= weight
α, β	= angle of attack and sideslip
$\delta a, \delta e, \delta r$	= aileron, elevator, and rudder deflection
ε	= thrust orientation angle
μ, γ, χ	= bank, flight path, and heading angles
π	= engine thrust setting
ϕ, θ, ψ	= roll, pitch, and yaw body Euler angles

I. Introduction

DYNAMICAL phenomena are often modeled by systems of differential, difference, or integral equations derived possibly from some scientific principle or physical insight. Dynamical systems theory is a branch of mathematics concerned with the properties common to these types of equations. A fairly well developed theory for linear dynamical systems exists and forms the basis of much of the mathematical analysis taught in the universities and practiced in industry. Unfortunately, in physical applications, the class of linear dynamical systems is an exception rather than the rule. Often nonlinear models are conveniently "linearized" using ad hoc assumptions. Analysis and design is then based on these simplified linear models. For flight at low rotational rates and small to moderate angles of attack and sideslip, aerodynamic forces and moments are predominantly linear functions. Hence classical linear longitudinal and lateral models are valid. At large angles of attack and sideslip and high rotational rates, nonlinearities in the kinematics and aerodynamic forces and moments become significant and it is these nonlinear characteristics that determine the pre- and postcritical behavior of the aircraft. Clearly a linear analysis is inapplicable for this regime. Indeed cases of nonlinear phenomena such as roll divergence, wing-rock, nose-slice, stall/spin, poststall departure, and other instabilities have been reported.[1–6] Analytical methods suggested for studying these nonlinear phenomena range from perturbation expansions in grams per volt,[7] pseudo-steady analysis where weight components are neglected,[8] projection techniques,[9–11] averaging, and normal forms to extract the critical modes.[12] For a computational approach, continuation algorithms applied to the full six-degree-of-freedom (DOF) system may be used to generate bifurcation diagrams.[13] In the following section the full nonlinear equations of motion for flight dynamics are stated together with relevant concepts from dynamical systems theory. This is followed by examples of nonlinear bifurcation analysis on reduced-order systems containing the critical dynamics. The utility of the reduced-order systems is then verified with the time response from a full six-DOF simulation. Robustness of the bifurcations with respect to unmodeled dynamics is also discussed.

II. Equations of Motion: Six-DOF Model

The classical longitudinal and lateral models for a symmetric, rigid aircraft are derived from a system of nine coupled, nonlinear ordinary differential equations (ODEs) for three translational modes $\{\alpha, \beta, V_T\}$, three rotational modes, and three Euler angles $\{\phi, \theta, \psi\}$ or $\{\gamma, \mu, \chi\}$. The nonlinear equations are as follows:

Translational motion:

$$\alpha' = q - (r\sin\alpha + p\cos\alpha)\tan\beta - g\{[T\sin(\alpha+\varepsilon) + L]/W - c_1\}/(V_T\cos\beta) \tag{1a}$$

$$\beta' = p\sin\alpha - r\cos\alpha + g\{[Y\cos\beta - (T\cos(\alpha+\varepsilon) - D) \times \sin\beta]/W - c_2\}/V_T \tag{1b}$$

$$V_T' = g(\{[T\cos(\alpha+\varepsilon) - D]\cos\beta + Y\sin\beta\}/W - c_3) \tag{1c}$$

where

$$\begin{aligned} c_1 &= \cos(\theta-\alpha) - (1-\cos\phi)\cos\theta\cos\alpha \\ &= \cos\mu\cos\gamma \\ c_2 &= \sin(\theta-\alpha)\sin\beta + \cos\theta\sin\phi\cos\beta \\ &\quad + (1-\cos\phi)\cos\theta\sin\alpha\sin\beta \\ &= \sin\mu\cos\gamma \\ c_3 &= \sin(\theta-\alpha)\cos\beta + \cos\theta\sin\phi\sin\beta \\ &\quad + (1-\cos\phi)\cos\theta\sin\alpha\cos\beta \\ &= \sin\gamma \end{aligned}$$

Rotational motion:

$$I_x p' + (I_z - I_y)qr - I_{xz}(r' + pq) = l \tag{2a}$$

$$I_y q' + (I_x - I_z)pr - I_{xz}(r^2 - p^2) = m \tag{2b}$$

$$I_z r' + (I_y - I_x)pq - I_{xz}(p' - qr) = n \tag{2c}$$

Received Aug. 2, 1993; revision received Dec. 21, 1993; accepted for publication March 7, 1994. Copyright © 1994 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

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Body axes orientation:

$$\phi' = p + (q \sin\phi + r \cos\phi) \tan\theta \tag{3a}$$

$$\theta' = q \cos\phi - r \sin\phi \tag{3b}$$

$$\psi' = (q \sin\phi - r \cos\phi) \sec\theta \tag{3c}$$

Or flight path axes orientation:

$$\mu' = p_w + (q_w \sin\mu + r_w \cos\mu) \tan\gamma \tag{4a}$$

$$\gamma' = q_w \cos\mu - r_w \sin\mu \tag{4b}$$

$$\chi' = (q_w \sin\mu - r_w \cos\mu) \sec\gamma \tag{4c}$$

with

$$p_w = (p \cos\alpha + r \sin\alpha) \cos\beta + (q - \alpha') \sin\beta$$

$$q_w = (-p \cos\alpha + r \sin\alpha) \sin\beta + (q - \alpha') \cos\beta$$

$$r_w = -p \sin\alpha + r \cos\alpha + \beta'$$

In general the aerodynamic forces $\{L, D, Y\}$, moments, $\{l, m, n\}$, and thrust T are nonlinear tabular functions of $\{\alpha, \beta, \alpha', \beta', h, M, p, q, r\}$ and the control settings $\{\delta a, \delta e, \delta r, \pi\}$. For the subsequent nonlinear (bifurcation) analysis it is necessary that explicit function approximations be found for the section of the tabular database where instabilities are expected to occur.

The analysis procedure for nonlinear ODEs consists of four main steps. This procedure provides a rigorous basis for stability analysis and avoids ad hoc assumptions like the small-angle approximations ($\sin\alpha \approx 0,\ \cos\alpha \approx 1$). Assume as given a system of ODEs in the form

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{5}$$

where $\boldsymbol{x}$ is a vector of state variables and $\boldsymbol{\lambda}$ represents the system parameters.

Step 1. Find the fixed (equilibrium) points $(\boldsymbol{x}_0, \boldsymbol{\lambda}_0)$ of the system:

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{\lambda}_0) \tag{6}$$

Step 2. Determine the stability of the fixed points from the eigenvalues of the Jacobian matrix $\boldsymbol{J}$ (linear analysis):

$$\boldsymbol{J} = \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]_{x=x_0, \lambda=\lambda_0} \tag{7}$$

Provided the Jacobian matrix is nonsingular, this step also provides a linear model valid for small perturbations $\Delta\boldsymbol{x}$ about the fixed point $(\boldsymbol{x}_0, \boldsymbol{\lambda}_0)$[14]:

$$\Delta\boldsymbol{x}' = [\boldsymbol{J}(\boldsymbol{x}_0, \boldsymbol{\lambda}_0)]\Delta\boldsymbol{x} \tag{8}$$

Step 3. If the system changes its stability, identify potential bifurcation points:

$$\det\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = 0 \tag{9}$$

Step 4. Determine the postcritical behavior using bifurcation theory (nonlinear analysis):

$$\Delta\boldsymbol{x}' = [\boldsymbol{J}]\Delta\boldsymbol{x} + \Delta\boldsymbol{x}^T[\boldsymbol{J}_2]\Delta\boldsymbol{x} + \cdots \tag{10}$$

For flight dynamics, step 1 is related to the problem of trimming the aircraft; i.e., for different altitudes and air speeds, find suitable control settings $\{\delta a, \delta e, \delta r, \pi\}$ so that no net forces and moments act on the aircraft.

Step 2 is concerned with the flying qualities of the aircraft. For example, for piloted military aircraft; MIL-F-8785C specifies acceptable values for these eigenvalues. This step also provides a mathematically correct linearization and emphasizes that the linear model is local (restricted to a neighborhood of the trim point).

Step 3 deals with the situation when the trim point is on the verge of instability (critical). This is indicated by eigenvalues crossing the imaginary axes in the complex plane. The states and parameter values at which the crossing occurs specify the bifurcation point. Typically the loss of stability occurs at large angles of attack and /or sideslip. When this happens, the linear model is no longer applicable since the aircraft will not remain within the local neighborhood of the trim point.

Whether the aircraft will settle into a new equilibrium after a bifurcation requires study of the nonlinear terms. In step 4 techniques from bifurcation theory can be used to determine the postcritical behavior without numerical simulation. Techniques like the Lyapunov–Schmidt method, the center manifold theorem, normal forms, and the method of averaging decompose the system into stable and critical modes often with a reduction in the dimension of the problem.[15–18] In the following examples, a simpler and more intuitive approach is used to extract the essential dynamics of the system. The utility of these simple models in predicting nonlinear phenomena such as pitch-up, wing-rock, nose-slice, and roll/yaw divergence is then investigated.

III. Nonlinear Analysis Tools

For this investigation the Hopf bifurcation theorem form dynamical systems theory is required.[17] The theorem is concerned with a loss of stability characterized by the crossing of a pair of complex eigenvalues. Consider the two-dimensional oscillatory system

$$x' = \varepsilon\lambda(\mu)x - \omega y + f(x, y) \tag{11}$$

$$y' = \omega x + \varepsilon\lambda(\mu)y + g(x, y) \tag{12}$$

where $\lambda(\mu_0) = 0$ at the bifurcation (critical) point and $f(x, y)$, $g(x, y)$ consists of nonlinear terms in x and y. Using a series of near-identity transformations, the theorem proves that close to the bifurcation point the amplitude of the oscillation, r, is governed by

$$r' = [\varepsilon\lambda(\mu) + ar]r^2 \tag{13}$$

The coefficient a is related to the system nonlinearities as follows:

$$a = \tfrac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + [f_{xy}(f_{xx} + f_{yy})$$
$$- g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]/16\omega \tag{14}$$

and the partial derivatives are all evaluated at the bifurcation point. Equation (13) is referred to as the normal form for the Hopf bifurcation. The postcritical behavior [i.e., $\lambda(\mu) > 0$] is either 1) sustained oscillation or 2) divergent oscillation, depending on whether the coefficient a is negative or positive, respectively.

In the event that additional stable modes are present, it can be shown rigorously that these stable states in the critical equations may be ignored.[18] This corresponds to the intuitive notion that the stable variables will ultimately decay and hence will not contribute to the dynamics of the critical modes. Extraction of the critical modes from an nth-dimensional system and computation of the normal form is best accomplished using a symbolic manipulator.[19]

IV. Longitudinal Dynamics

For the first example, a simplified model of the longitudinal dynamics will be used. It will be assumed that $\beta = p = r = \delta a = \delta r = 0$ and that the thrust setting is adjusted for level flight at constant air speed. The flight dynamics is then governed by a nonlinear short-period approximation:

$$\alpha' = q - g[(T \sin\alpha + q_T S C_L)/W - 1]/V_T \tag{15}$$

$$q' = q_T S c C_m / Iy \tag{16}$$

where the lift and pitching moment coefficients C_L, C_m are nonlinear functions of α, q, and δe. The function approximations obtained are

$$C_L = \alpha[4.4575 + \alpha(-10.5922\alpha - 0.556634\delta e)] + 1.02302\delta e \tag{17}$$

$$C_m = 0.0170477 + \alpha(-0.333791 + 0.99557\delta e)$$
$$- 1.647\delta e - 0.075q \tag{18}$$

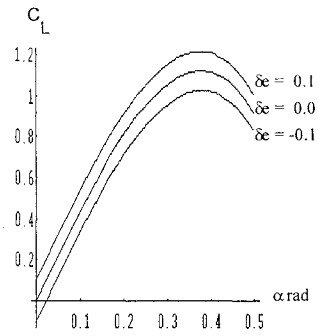

Fig. 1 Lift coefficient.

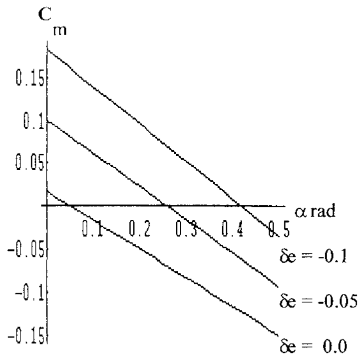

Fig. 2 Static pitching moment coefficient.

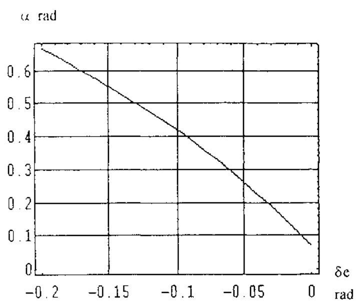

Fig. 3 Fixed-point angle of attack α as function of elevator setting.

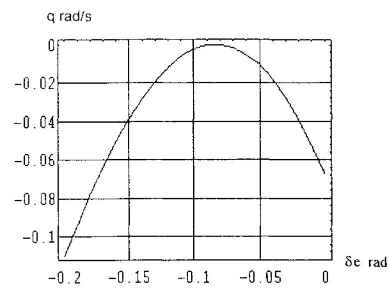

Fig. 4 Fixed-point pitch rate q as function of elevator setting.

These coefficients are shown in Figs. 1 and 2. The fixed points are obtained by solving $\alpha' = 0$, $q' = 0$, for different elevator settings. These are plotted in Figs. 3 and 4. The stability of the fixed points are then determined by the eigenvalues of the Jacobian matrix $\boldsymbol{J}$. Figure 5 shows the real part of the complex eigenvalues as a function of elevator setting. It also indicates that a loss of stability will occur for $-0.12 < \delta e < -0.11$. A more precise location of the bifurcation point can be obtained by solving the equation $T[J] = 0$ together with $\alpha' = 0$ and $q' = 0$ for $\alpha, q, \delta e(\alpha = 0.463, q = -0.009, \delta e = -0.115)$. This loss of stability that occurs through the crossing of a complex pair of eigenvalues indicates the presence of a Hopf bifurcation. Physically this means that sustained or divergent oscillations in α and q will be observed. The postcritical behavior (for $\delta e > -0.115$) is now determined by the Hopf bifurcation theorem. Bringing the equations to normal form yields a

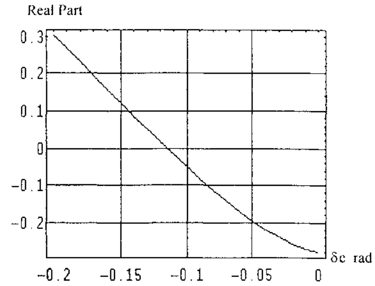

Fig. 5 Bifurcation diagram: real part of short period eigenvalues as function of elevator setting.

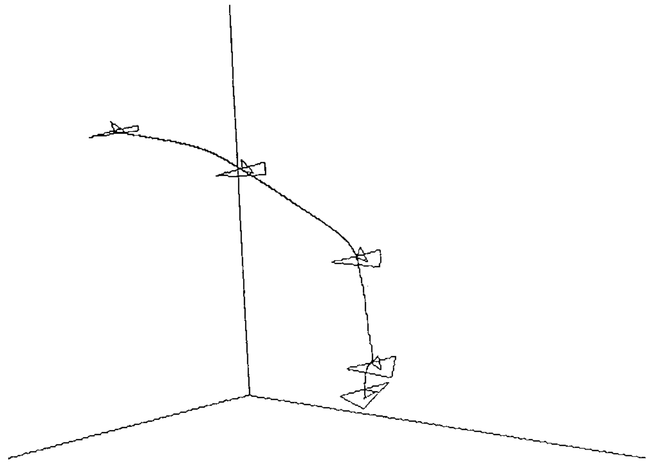

Fig. 6 Postbifurcation flight trajectory for $\delta e = -0.12$ rad.

positive cubic coefficient ($a \approx 0.37$). Hence, according to the Hopf bifurcation theorem, the oscillations will be divergent.

Deflection of the elevators can cause the aircraft to pull up or push over into a loop. This will lead to changes in altitude and air speed. Hence it is necessary that we examine the robustness of the Hopf bifurcation, i.e., whether results obtained from the reduced-order approximation carry over to the full six-DOF model. In Fig. 6 the six-DOF trajectory just after bifurcation ($\delta e = -0.12$ rad) is shown. It can be observed that the divergent oscillations in α have led to stall with a sharp loss of altitude. Thus a reduced-order nonlinear analysis provides a useful indication of the six-DOF dynamics.

V. Lateral Dynamics

In the second example bifurcation theory will be used to detect the presence of an oscillatory instability call wing-rock. Assuming that the flight vehicle is trimmed for level flight at constant air speed, i.e., $q = \alpha' = h' = V_T' = 0$, the governing equations are

$$\beta' = p \sin\alpha - r\cos\alpha + g[Y/(W\cos\beta) + \sin\mu]/V_T \tag{19}$$

$$I_x p' - I_{xz} r' = l \tag{20}$$

$$I_z r' - I_{xz} p' = n \tag{21}$$

$$\mu' = (p\cos\alpha + r\sin\alpha)\cos\beta \tag{22}$$

Note that typically it is further assumed that $\mu \approx \phi$, $\phi' = p$, and the small-angle approximations $\sin\alpha \approx 0$ and $\cos\alpha \approx 1$ are used. For large angles of attack and sideslip these assumptions are not applicable. The aerodynamic model obtained by functional approximation for the chosen flight conditions ($\delta a = \delta r = 0$) takes the form

$$\begin{gathered} Y = q_T S C_Y \\ C_Y = -\beta(0.26 + 0.22\alpha + 4.28\beta^2) + C_{Y_p}(\alpha)P \end{gathered} \tag{23}$$

$$\begin{gathered} l = q_T S b C_l \\ C_l = -1.14\alpha\beta + C_{lp}(\alpha)P + C_{lr}(\alpha)R \end{gathered} \tag{24}$$

$$\begin{gathered} n = q_T S b C_n \\ C_n = \beta(0.14 + 0.52\alpha + 5.4\beta^2) + C_{np}(\alpha)P + C_{nr}(\alpha)R \end{gathered} \tag{25}$$

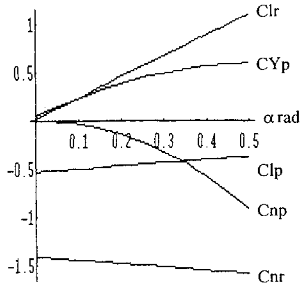

Fig. 7 Lateral dynamic derivatives as function of angle of attack.

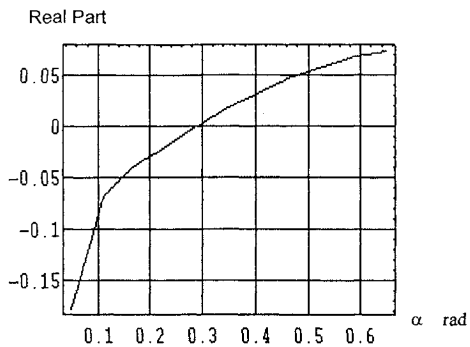

Fig. 8 Bifurcation diagram: real part of Dutch roll eigenvalues as function of angle of attack.

The dependence of the dynamic stability derivatives on the angle of attack is shown in Fig. 7. The magnitude of the roll damping derivative C_{lp} decreases with angle of attack and is a likely cause of instability. It is easily verified that the origin ($\beta = p = r = \mu = 0$) is a fixed point. The variation of the real part of the complex Dutch roll eigenvalues with the angle of attack is shown in Fig. 8. The Dutch roll mode becomes unstable at $\alpha \approx 0.289$. The postcritical behavior can be determined by ignoring the stable roll and spiral modes and applying the Hopf bifurcation theorem to the Dutch roll modes alone. The normal-form computation yields $a \approx -0.0876$, indicating that sustained oscillations (wing-rock) will be present for $\alpha > 0.289$. For the six-DOF simulation, trimmed level flight was first achieved before a 0.1-rad perturbation in the sideslip angle was applied. Unlike the previous example a gradual loss of stability with sustained oscillation in the roll rate and angle was observed for $\alpha > 0.29$ rad.

VI. Conclusion

In this study it is shown that a nonlinear (bifurcation) analysis may be carried out on reduced-order models derived from physical insight. This approach avoids the involved algebraic manipulations required by direct application of the averaging theorem, the center manifold theorem, and other similar methods to the full six-DOF system. Furthermore the bifurcations predicted by the reduced-order method were found to be robust to unmodeled dynamics of the stable modes. Useful guidelines to the six-DOF nonlinear dynamics can thus be derived without extensive parameter sweeps/numerical simulation or the requisite computation associated with continuation methods.

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